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PROOF OF THE LAW OF DIMINISHING RETURNS

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ABSTRACT

Based on a general mathematical model of a technology, implying certain properties for the production function, weak and strong forms of a physical law of diminishing returns are derived. It is also shown that the classical forms of this law hold if the technology is homogeneous (degree one) and the production possibility sets of the technology are strictly convex, but the latter property violates an essential property of a technology, namely that these sets have bounded efficient subsets.

PROOF OF THE LAW OF DIMINISHING RETURNS†

by

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1. INTRODUCTION

For 200 years, since it was first expressed (for land) by the French economist Turgot (1767), [13], a law of diminishing returns in the physical output of production has played a central role in the marginal analysis of economic theory, stating in some fashion that the output from production will eventually suffer decreasing increments or decreasing average return if the inputs of some factors of production are fixed and the others are increased indefinitely by some equal increments. Divorced of its reference solely to agriculture, diminishing returns are taken as a fundamental law for technology to support economic theories of equilibrium and price determination.

Thirty three years ago, in two papers published in the Zeitschrift für Nationalökonomie [6] and subsequently re-issued in english in Economic Activity Analysis (1954, edited by Oskar Morgenstern), K. Menger gave a penetrating, albeit entertaining, discussion showing that there has been considerable confusion in the statements of the law and the arguments adduced for it, involving such emminent classical economists as Wicksell (1909), Boehm-Bawerk (1912) and L. V. Mises (1933). In recent times, well-known economists (e.g., Samuelson in Economics, McGraw Hill, 1964) refer to the proposition as a "fundamental law of economics and technology" on the one hand, and also describe it as an "important, often-observed, economic and technical regularity," implying on the other hand that it is not a law but a statistical phenomena.

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†Dedicated to my friend Oskar Morgenstern who called this issue to my attention and urged me to work on it.

With the advent of the notion of a production function (circa 1910), deductions (explanations) of the law have followed from mathematical properties assumed for the production function, and most recently by Eichhorn [5] in the Zeitschrift für Nationalökonomie. Since the law of diminishing returns is a statement concerning technology, from which the production function is a derived concept, a study of the logical relationship between statements of the law and basic concepts in the theory of production should start with a definition of a technology.

A technology is given precise mathematical definition as a family of sets $T : L(u)$, $u \in [0,+\infty)$ in the nonnegative domain of an n-dimensional Euclidian space, with certain properties which are presumed to be generally applicable. The members of this family are indexed by a real, nonnegative variable u , denoting output rate, and each set $L(u)$ specifies the set of input vectors $x = (x_1,x_2, \ldots, x_n)$ yielding at least the output rate u . The production function $\Phi(x)$ of the technology is then defined on this family of sets for an input vector x as the maximal output rate obtainable with x , giving to it the classical meaning, and the properties of the production function are derived from those of the sets $L(u)$. These formulations permit substitutions between the factors of production, both as alternative and complementary means of production. The substitutions of primary interest are those on the boundaries of the sets $L(u)$ which are technologically efficient, i.e., input vectors for an output u such that a decrease of any of the inputs without increasing an input will fail to produce the output rate u .

One important property (premise) for the input sets $L(u)$ in the definition of the technology is that the efficient subset for each value of u is bounded, i.e., technologically efficient production of an output rate u is not made with an input vector which has infinitely large application of any factor of production.

Strangely, the production functions in common use which exhibit diminishing returns, e.g., the Cobb-Douglas and CES functions, violate this property and strictly speaking they are not production functions.

In this conceptual framework, it is clear that diminishing returns are not obtainable by fixing the inputs of any arbitrarily chosen subset of the factors of production. For this reason, a definition of an essential combination of the factors of production is introduced, as one for which positive output cannot be obtained if these factors are not used in production, and it is premised that a technology has at least one essential combination of the factors of production. Then it is shown, by purely mathematical deduction from the general properties of a technology, that there exists a positive bound upon the inputs of the factors of an essential combination such that output is bounded when the inputs of the factors of the essential combination are restricted to this bound and the inputs of the remaining factors are increased indefinitely.

It is shown by counterexample satisfying the properties assumed for a technology that essentiality of a combination of the factors of production does not imply that output is bounded for *any positive bound* upon the inputs of the factors of an essential combination, when the inputs of the remaining factors are increased indefinitely. An essential combination is called strongly limitational if output is bounded for all positive bounds upon the inputs of this combination, i.e., unbounded output cannot be obtained under any bounded inputs for an essential combination.

Two weak forms of the law of diminishing returns, one for product increments and one for average return, are deduced for a technology. These expressions of the law are of the form described by Menger as "intersecting assertions," as distinguished from the traditional forms which imply strict concavity of the production function in sufficiently large variable inputs when the inputs of some factors are held fixed. Two corresponding strong forms of the law hold if an essential combination of the factors is strongly limitational.

The properties of a substructure (i.e., realization of a technology with positive bounds upon the inputs of a subset of the factors of production) are investigated. If the subset of the factors is not an essential combination, the resulting substructure is a technology when these bounds are zero, of more limited alternatives but nevertheless one with the same general properties, and no law of diminishing returns can be deduced.

The traditional forms of the law cannot be obtained without assumptions on the fine structure of a technology which are contrived to obtain the result. It is not infrequent in economic studies to assume that the production function is positively homogeneous of degree one, i.e., $\lambda L(u) = L(\lambda u)$ for the technology. It is shown that this assumption leads to nonincreasing returns over the whole range, i.e., not merely for sufficiently large variable inputs. If it is assumed further that the input sets are strictly convex for positive output, a proposition of diminishing returns is obtained over the whole range of variable inputs, but this assumption implies that the efficient subsets of the technology are unbounded and each factor of production is essential. These properties, i.e., homogeneity of degree one and strict convexity of the input sets $L(u)$ for $u > 0$, generalize the assumptions of Eichhorn and they are possessed by the Cobb-Douglas and CES production functions.

The term "proof" used in the title of this paper is intended to convey a sequence of logically valid statements for technologies defined by the input sets $L(u)$ with the properties stipulated.

The mathematical treatment of the structure of production used in this paper is an extension of the work of the author provided in (a) Unternehmensforschung, Vol 11, 1967, No. 4, "The Notion of a Production Function," (b) Theory of Cost and Production Functions, book manuscript submitted to Princeton University Press, June 1969.

2. DEFINITION OF A PRODUCTION TECHNOLOGY

A production technology consists of certain alternative means and arrangements by which goods or services are produced, not all possibilities of which need be realized in practice. The distinct goods and services which may be used as inputs are the factors of production, and free goods or services are not excluded, since the market price of a commodity has no bearing upon the technical roles of the factors of production. The technology exists independently of the political and social structure in which it may operate and also of the scarcity of the inputs, i.e., it is a blueprint for production.

It is assumed that a single good or service is obtainable as an output of the technology.[†] Let $u \in [0,+\infty)$ denote the output rate, and take $x = (x_1, x_2, \ldots, x_n)$ to denote the input rates of the factors of production with x restricted to the nonnegative domain of a Euclidian space R^n, denoted by R^n_+. It is not assumed that x must be strictly positive for u positive, i.e., some of the factors may be complete substitutes for others.

Definition 1: A production input set of the technology, denoted by $L(u)$, is the set of all input vectors $x \in R^n_+$ yielding at least the output rate $u \in [0,+\infty)$.

Clearly not all input vectors x belonging to an input set $L(u)$ are technologically efficient. Those which are efficient are given by the following definition:

Definition 2: The efficient subset $E(u)$ of an input set $L(u)$ is given by:

[†] The law classically refers to this situation.

$$E(u) = \{x \mid x \in L(u)\ ,\ y \leq x \Rightarrow y \notin L(u)\}\ .^{\dagger}$$

Then, a production technology is defined as follows:

<u>Definition 3</u>: A production technology is a family of production input sets $T : L(u)\ ,\ u \in [0,+\infty)$ satisfying:

P.1 $L(0) = R^n_+$, $0 \notin L(u)$ for $u > 0$.

P.2 $x \in L(u)$ and $x' \geqq x$ imply $x' \in L(u)$.

P.3 If (a) $x > 0$, or (b) $x \geq 0$ and there exists a real number $\bar{\lambda} > 0$ and output rate $\bar{u} > 0$ such that $(\bar{\lambda} \cdot x) \in L(\bar{u})$, the ray $\{\lambda \cdot x \mid \lambda \geqq 0\}$ intersects $L(u)$ for all $u \in [0,+\infty)$.

P.4 $u_2 \geqq u_1 \geqq 0$ implies $L(u_2) \subseteq L(u_1)$.

P.5 $\bigcap_{u \in [0,+\infty)} L(u)$ is empty.

P.6 $\bigcap_{0 \leqq u < u_o} L(u) = L(u_o)$ for $u_o > 0$.

P.7 $L(u)$ is closed for all $u \in [0,+\infty)$.

P.8 $L(u)$ is convex for all $u \in [0,+\infty)$.

P.9 $E(u)$ is bounded for all $u \in [0,+\infty)$.

P.10 For $\lambda \geqq 1$, $u \in [0,+\infty)$, $\lambda \cdot L(u) \subseteq L(\lambda u)$ and $\frac{1}{\lambda} L(u) \supseteq L\left(\frac{1}{\lambda} u\right)$.

The Properties P.1, ..., P.10 are taken as valid or generally acceptable for any technology. Property P.1 states merely that any nonnegative input vector yields at least null output, and positive output cannot be obtained from a

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† $y \geqq x$ means $y_i \geqq x_i$, $i = 1,2, \ldots, n$, including $y = x$.
$y \geq x$ means $y_i \geqq x_i$, $i = 1,2, \ldots, n$, excluding $y = x$.

null i ut vector. Property P.2 implies disposability of inputs. For example, if ch cal fertilizer is used as an input with land to produce a crop and exces e amounts of the former are available relative to land, in an input vecto x, not all of the fertilizer has to be actually used to decrease the crop. merely disposes of the excess fertilizer. Fortuitous events, such as flo d are not incompassed. Thus, the technology is regarded as a rational, contr able process.

P perty P.3 states first that any output rate $u \in [0,+\infty)$ can be realized by sca r magnification of a positive input vector, although not necessarily in an eff ient way, and second that, if a positive output rate can be obtained by scala agnification of a semi-positive input vector x, any null inputs of x are no required for production and the same attainability of all positive output rates lds by scalar magnification of the semi-positive input vector x. The family f input sets $L(u)$ defines the input unconstrained technical possibilities. Divisi lity of output rate is not implied, but disposability of outputs is assum

P perty P.4 is clearly appropriate, since an input vector x yielding an outpu ate u_2 also yields any output rate not exceeding u_2, and Property P.5 is mer y a precise way of stating that an unbounded output rate cannot be attained by a nded input vector.

perties P.6 and P.7 have only mathematical significance. Property P.6 is impos in order to guarantee the existence of the production function $\Phi(x)$ as the m num output rate attainable with an input vector x. Property P.7 is impos in o er to be able to define the production isoquant for an output rate u as subset of the boundary of the input set $L(u)$ relative to R^n.

perty P.8 is valid for time-divisibly-operable technologies. For example, i. x $L(u)$, $y \in L(u)$ and $\theta \in [0,1]$, the input vector $[(1 - \theta)x + \theta y]$ ay be int rpreted as an operation of the technology a fraction $(1 - \theta)$ of some

unit time interval with the input vector x and the remaining fraction θ with y, assuring at least the output rate u.[†] Nothing is implied about the efficiency of such an operation.

Property P.9 is imposed as an obvious physical fact that no output rate is attained efficiently (in a technological sense) by an unbounded input vector. This property, frequently ignored by the production functions in common use, is essential for the arguments to follow. It also assures, for any semi-positive price vector p for the inputs x, that an optimal input vector x^* can be realized to minimize costs. Note that free goods are not excluded from the inputs x.

Property P.10 is taken valid for the following reasons. If $x \in L(u)$ (i.e., x realizes at least the output rate u) and $\lambda \geqq 1$, then $(\lambda \cdot x)$ may realize at least the output rate (λu), merely by a time-divisible replication of the arrangement with x producing u, but $\lambda \cdot L(u)$ may be a proper subset of $L(\lambda \cdot u)$, since, if $x \in E(u)$ is an efficient input vector, $(\lambda \cdot x)$ may not be efficient for $L(\lambda \cdot u)$. For the same reasons, if $x \in L\left(\frac{u}{\lambda}\right)$ and $\lambda \geqq 1$, then $(\lambda \cdot x)$ will yield at least the output rate u, whence $x \in \frac{1}{\lambda} L(u)$ and $L\left(\frac{u}{\lambda}\right) \subset \frac{1}{\lambda} L(u)$.[††]

In the foregoing definition of a technology, nothing is assumed which is peculiar to any particular physical system of production. One consequence of Property P.10 is that, with no limitations on the inputs of the factors of production, average costs are nonincreasing with respect to scale of output. For $u > 0$, the minimal cost rate of production for a price vector p of x is

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[†]Indeed, the input $[(1 - \theta)x + \theta y]$ may have no meaning unless so interpreted.

[††]Diminishing returns from extensive application of a factor like land does not contradict Property P.10, since we are concerned with the unconstrained technical alternatives.

$$Q(u,p) = \min_{x} \{p \cdot x \mid x \in L(u)\} ,$$

and, for $\lambda \geq 1$, Property P.10 implies

$$\begin{aligned} Q(\lambda u,p) &= \min_{x} \{p \cdot x \mid x \in L(\lambda u)\} \\ &\leq \min_{x} \{p \cdot x \mid x \in \lambda L(u)\} \\ &= \lambda \min_{x} \left\{p\left(\frac{x}{\lambda}\right) \middle| \frac{x}{\lambda} \in L(u)\right\} \\ &= \lambda\, Q(u,p) , \end{aligned}$$

whence

$$\frac{Q(\lambda u,p)}{\lambda \cdot u} \leq \frac{Q(u,p)}{u} ,$$

for any $\lambda \geq 1$ and $u > 0$.

3. THE PRODUCTION FUNCTION FOR A TECHNOLOGY

The production function $\Phi(x)$ for a technology $T : L(u)$, $u \in [0,+\infty)$ is defined by

$$\text{(1)} \qquad \Phi(x) = \text{Max } \{u \mid x \in L(u)\} \text{ , } x \in R^n_+ \text{ ,}$$

as the maximum output rate attainable with an input vector x .

The existence of the production function $\Phi(x)$ and its properties implied by Properties P.1, ..., P.8 of the technology T are proved in [11] and [12], but will be repeated here briefly for completeness of discussion. Also, the properties of $\Phi(x)$ implied by the Properties P.9 and P.10 will be stated and proved.

Let $x \geqq 0$ be arbitrary. Then $x \in L(0)$ due to Property P.1. Also there exists a finite output rate u' such that $x \notin L(u)$ for $u \geqq u'$, due to Properties P.4 and P.5. Consequently,

$$\text{Sup } \{u \mid x \in L(u)\} = u_o \quad \text{(finite)} \text{ ,}$$

and $x \in L(u)$ for $u \in [0,u_o)$. But, due to Property P.6, $\{u \mid x \in L(u)\} = [0,u_o]$. Hence $\Phi(x)$ exists for any $x \in R^n_+$ and $\Phi(x)$ is finite for any bounded input vector x . Thus, the existence of the production function $\Phi(x)$ with $\Phi(0) = 0$ follows only from Properties P.1, P.4, P.5 and P.6 of the technology T . Property P.2 implies $\Phi(x') \geqq \Phi(x)$ for any $x' \geqq x$, since $\{u \mid x \in L(u)\} \subset \{u \mid x' \in L(u)\}$. Property P.3 implies: (a) if $x > 0$, $\Phi(\lambda x) \to +\infty$ as $\lambda \to +\infty$, (b) if $x \geq 0$ and $\Phi(\bar{\lambda} \cdot x) > 0$ for some $\bar{\lambda} > 0$, then $\Phi(\lambda x) \to +\infty$ as $\lambda \to +\infty$. Property P.7 implies $\Phi(x)$ is upper semi-continuous, because $L(u) = \{x \mid \Phi(x) \geqq u\}$ is closed for $u \geqq 0$ and $\{x \mid \Phi(x) \geqq u\} = R^n_+$ (closed) for $u < 0$, and the closure of the level sets $\{x \mid \Phi(x) \geqq u\}$ of $\Phi(x)$ for all $u \in (-\infty,+\infty)$ is an if and only if condition for the upper semi-continuity of $\Phi(x)$.

Property P.8 implies that the production function is quasi-concave, i.e., for $x \geqq 0$, $y \geqq 0$ and $\theta \in [0,1]$, $\Phi((1-\theta)x + \theta y) \geqq \text{Min}\,[\Phi(x),\Phi(y)]$, because letting $\tau = \text{Min}\,[\Phi(x),\Phi(y)]$, $x \in L(\tau)$, $y \in L(\tau)$ and the convexity of $L(\tau)$ implies $[(1-\theta)x + \theta y] \in L(\tau)$.

For the implication of Property P.9, suppose $\Phi(x) = u$. Then $\Phi(x) = u$ with $\Phi(y) < u$ for $y \leq 0$ on the bounded subset $E(u)$ of $L(u)$. If $\Phi(x) = u$ and $x \notin E(u)$, there exists a point $y \leq x$ such that $\Phi(y) \geqq u$. Hence, $\Phi(x)$ takes a constant value, with $\Phi(y) < \Phi(x)$ for $y \leq x$, on bounded subsets of R^n_+.

Property P.10 implies that the production function $\Phi(x)$ is super-homogeneous, i.e., for $\lambda \geqq 1$ $\Phi(\lambda x) \geqq \lambda\Phi(x)$ and $\Phi\left(\frac{x}{\lambda}\right) \leqq \frac{1}{\lambda}\Phi(x)$, because if $u = \Phi(x)$ then $x \in L(u)$ and $(\lambda x) \in L(\lambda u)$ implying $\Phi(\lambda x) \geqq \lambda\Phi(x)$, while if $\frac{u}{\lambda} = \Phi\left(\frac{x}{\lambda}\right)$ then $\frac{x}{\lambda} \in L\left(\frac{u}{\lambda}\right)$ and $x \in L(u)$ implying $\Phi(x) \geqq u = \lambda\Phi\left(\frac{x}{\lambda}\right)$.

Finally, Properties P.8 and P.10 imply that the production function is super-additive, i.e., if $x \in R^n_+$, $y \in R^n_+$ then $\Phi(x+y) \geqq \Phi(x) + \Phi(y)$. To prove this statement, let $\Phi(x) > 0$ and $\Phi(y) > 0$ and let $u = \text{Max}\,[\Phi(x),\Phi(y)]$. Then, since Property P.10 implies that $\Phi(x)$ is super-homogeneous,

$$\Phi\left(\frac{u}{\Phi(x)}\cdot x\right) \geqq u\ ,\quad \Phi\left(\frac{u}{\Phi(y)}\cdot y\right) \geqq u\ ,$$

implying that the input vectors $\frac{u}{\Phi(x)}\cdot x$, $\frac{u}{\Phi(y)}\cdot y$ belong to $L(u)$. The convexity of $L(u)$, i.e., property P.8, then implies that

$$\Phi\left((1-\theta)\,\frac{u}{\Phi(x)}\cdot x + \theta\frac{u}{\Phi(y)}\cdot y\right) \geqq u\ .$$

Take $\theta = \dfrac{\Phi(y)}{\Phi(x) + \Phi(y)}$ and use the super-homogeneity of $\Phi(x)$ to obtain

$$\frac{\Phi(x+y)}{\Phi(x) + \Phi(y)}\cdot u \geqq \Phi\left(\frac{u}{\Phi(x) + \Phi(y)}\cdot(x+y)\right) \geqq u\ ,$$

whence $\Phi(x + y) \geqq \Phi(x) + \Phi(y)$. If either $\Phi(x)$ or $\Phi(y)$ or both are zero, the same inequality holds.

In summary, the following proposition holds:

<u>Proposition 1</u>: The production function $\Phi(x)$ defined by (1) on the technology $T : L(u)$, $u \in [0,+\infty)$ has the following properties:

A.1 $\Phi(0) = 0$.

A.2 $\Phi(x)$ is finite for bounded $x \in R_+^n$.

A.3 $\Phi(x') \geqq \Phi(x)$ for $x' \geqq x$.

A.4 If (a) $x > 0$, or (b) $x \geq 0$ and $\Phi(\bar{\lambda} \cdot x) > 0$ for some $\bar{\lambda} > 0$, $\Phi(\lambda x) \to +\infty$ as $\lambda \to +\infty$.

A.5 $\Phi(x)$ is upper semi-continuous in $x \in R_+^n$.

A.6 $\Phi((1 - \theta)x + \theta y) \geqq \text{Min}\, [\Phi(x),\Phi(y)]$ for x , $y \in R_+^n$, $\theta \in [0,1]$.

A.7 $\Phi(\lambda x) \geqq \lambda\Phi(x)$ and $\Phi\left(\frac{x}{\lambda}\right) \leqq \frac{1}{\lambda}\, \Phi(x)$ for $x \in R_+^n$ and $\lambda \geqq 1$.

A.8 $\Phi(x)$ takes constant values, with $\Phi(y) < \Phi(x)$ for $y \leq x$, on bounded subsets of R_+^n .

A.9 $\Phi(x + y) \geqq \Phi(x) + \Phi(y)$ for x , $y \in R_+^n$.

Note that the technology T and its related production function $\Phi(x)$ are expressions of the unconstrained technological alternatives. Restrictions on the input vectors x are to be handled separately and not incorporated into the definition of the production function $\Phi(x)$.

One consequence of Property A.9 is that the unconstrained minimum cost of yielding at least an output rate u by combining two operations of a technology is equal to or less than the sum of the minimal costs for the two separate operations

to yield in at least the same output rate, since

$$\{(x + y) \mid \Phi(x + y) \geq u\} \supset \{(x + y) \mid \Phi(x) + \Phi(y) \geq u\} ,$$

and

$$\operatorname*{Min}_{x,y} \{p\cdot(x + y) \mid \Phi(x + y) \geq u\} \leq \operatorname*{Min}_{x,y} \{p\cdot(x + y) \mid \Phi(x) + \Phi(y) \geq u\} .$$

If a production function $\Phi(x)$ is given, the level sets of this function satisfy the following proposition.

<u>Proposition 2</u>: The level sets $L_\Phi(u) = \left\{x \mid \Phi(x) \geq u , x \in R^n_+\right\}$, $u \in [0,+\infty)$ for a nonnegative single valued real function $\Phi(x)$ defined on R^n_+ with the Properties A.1, ..., A.9 possess the Properties P.1, ..., P.10, and the production function defined on the sets $L_\Phi(u)$ is identical to $\Phi(x)$.

A proof of Proposition 2 is given in [11] and [12] for the Properties A.1, ..., A.6 and P.1, ..., P.8, and the extension to A.7, A.8, A.9 and P.9, P.10 is direct.

It is convenient at this point to introduce a definition of the production isoquant for an output rate u , and to state two propositions concerning the efficient subsets $E(u)$ of the production isoquants.

<u>Definition 4</u>: The production isoquant for an output rate $u \in [0,+\infty)$ is the boundary of the set $L(u)$, excluding those points which are not at minimal ray distance from the origin.

If part of the boundary of $L(u)$ relative to R^n coincides with the boundary of R^n_+ , those points x on such commonpart for which $(\lambda\cdot x) \in L(u)$ for $\lambda < 1$ are not included in the production isoquant, because, macroscopically, they cannot

be efficient input vectors.

Two propositions hold for the efficient subsets $E(u)$:

<u>Proposition 3</u>: $E(u)$ is nonempty for all $u \in [0,+\infty)$.

<u>Proposition 4</u>: $L(u) = \bar{E}(u) + R_+^n$, $u \in [0,+\infty)$.†

The proofs of these two propositions are given in [11] and [12]. One cannot conclude that $E(u)$ is closed--see the counter example in [1]--and, although $L(u) = E(u) + R_+^n$, it is sufficient for our purposes to work with the closure $\bar{E}(u)$ of $E(u)$.

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† The sum $A + B$ of two sets A and B in R_+^n is the set of points of the form $(x + y)$ where $x \in A$ and $y \in B$.

4. LIMITATIONAL FACTORS OF PRODUCTION

The general definition of a technology given in Section 2 permits substitutions between the factors of production to attain efficiently any given output rate u (i.e., the alternatives $E(u)$) and it is not assumed that positive inputs of any particular factor of production or combination of factors are required for positive output, nor that a positive bound upon the inputs of a factor or combination of factors limits the output which may be realized under increasing applications of the other factors. In a word, we have been concerned with the unconstrained alternatives of a technology.

For the investigation of a law of diminishing returns we must turn our attention to the possible limitational character of the factors of production.

Let

$$D_1 = \{x \mid x > 0 \ , \ x \in R^n\}$$

$$D_2 = \left\{x \mid x \geq 0 \ , \ \prod_1^n x_i = 0 \ , \ x \in R^n\right\} .$$

Then

$$R_+^n = \{0\} \cup D_1 \cup D_2 \ .$$

The boundary points of R_+^n relative to R^n excluding the null input vector, i.e., D_2, are classified further by

$$D_2(\nu_1, \nu_2, \ldots, \nu_k) = \left\{x \mid x \in D_2 \ , \ x_{\nu_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, k\right\}$$

for integers k such that $1 \leqq k \leqq (n - 1)$.

<u>Definition 5</u>: The combination $(\nu_1, \nu_2, \ldots, \nu_k)$, $1 \leq k \leq n-1$, is essential if and only if $D_2(\nu_1, \ldots, \nu_k) \cap L(u)$ is empty for all $u > 0$, or equivalently $\Phi(x) = 0$ for all $x \in D_2(\nu_1, \ldots, \nu_k)$.

Two propositions clarify this definition of the essentiality of a combination of the factors of production.

<u>Proposition 5</u>: If $x \in$ interior of $D_2(\nu_1, \ldots, \nu_k)$, i.e., $x_{\nu_i} = 0$ for $i \in \{1,2, \ldots, k\}$ and $x_{\nu_j} > 0$ for $j \notin \{1,2, \ldots, k\}$, and $\Phi(\lambda x) = 0$ for $\lambda \in [0,+\infty)$, then $\Phi(y) = 0$ for all $y \in D_2(\nu_1, \nu_2, \ldots, \nu_k)$ and the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is essential.

<u>Proposition 6</u>: If $x \in$ interior of $D_2(\nu_1, \nu_2, \ldots, \nu_k)$ and $\Phi(\bar{\lambda} \cdot x) > 0$ for some scalar $\bar{\lambda} > 0$, then for all input vectors $y \in$ interior of $D_2(\nu_1, \ldots, \nu_k)$ there exists a positive scalar λ_y such that $\Phi(\lambda_y \cdot y) > 0$.

Propositions 5 and 6 follow directly from Properties A.3 and A.4 of the production function $\Phi(x)$.

Thus, either a combination $(\nu_1, \nu_2, \ldots, \nu_k)$ of the factors of production is essential by $\Phi(x) = 0$ for all $x \in D_2(\nu_1, \nu_2, \ldots, \nu_k)$, or for all $y \in$ interior of $D_2(\nu_1, \nu_2, \ldots, \nu_k)$ there exists a positive scalar λ_y such that positive output may be obtained with the input $(\lambda_y \cdot y)$. If a combination $\left(\nu_1, \nu_2, \ldots, \nu_{k_o}\right)$ is essential, clearly any combination $(\nu_1, \nu_2, \ldots, \nu_k)$ for $k_o \leq k \leq (n-1)$ is likewise essential. The combination consisting of all factors of production is obviously essential by virtue of Properties P.1 and A.1, but this does not imply that any factor or lesser combination of factors is essential.

In fact, the Properties P.1, ..., P.10 used in Section 2 to define a technology do not imply the existence of an essential combination $(\nu_1, \nu_2, \ldots, \nu_k)$ for $1 \leqq k \leqq (n-1)$.

Now suppose for a combination $(\nu_1, \nu_2, \ldots, \nu_k)$ of the factors of production that a positive bound $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ is imposed on the input of these factors. How may this bound impose a limitation (if at all) upon the outputs which may be obtained under unrestricted application of the other factors? For the investigation of this question two definitions are introduced:

<u>Definition 6</u>: A combination $(\nu_1, \nu_2, \ldots, \nu_k)$ of the factors of production is Weak Limitational if there exists a positive bound $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$ such that $\Phi(x)$ is bounded for $x \geqq 0$ and $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$.

<u>Definition 7</u>: A combination $(\nu_1, \nu_2, \ldots, \nu_k)$ of the factors of production is Strong Limitational if, for all positive subvectors $\left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$, $\Phi(x)$ is bounded for $x \geqq 0$ and $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, x^o_{\nu_2}, \ldots, x^o_{\nu_k}\right)$.

Clearly, if a combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is strong limitational it is weak limitational. Also, it would seem that if a combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is not essential it would not be limitational in either sense. To pursue this issue, suppose that a combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is not essential. Then, it follows from Proposition 6 that for all input vectors $y \in$ interior of $D_2(\nu_1, \nu_2, \ldots, \nu_k)$ there exists a positive scalar λ_y such that $\Phi(\lambda_y \cdot y) > 0$, and it follows from

Property A.4 of the production function that, for any input vector $y \in$ interior of $D_2(\nu_1, \nu_2, \ldots, \nu_k)$, $\Phi(\lambda y)$ is unbounded for $\lambda \in [0, +\infty)$. Consequently, due to Property A.3, the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is not weak limitational, and the following proposition holds:

<u>Proposition 8</u>: A combination $(\nu_1, \nu_2, \ldots, \nu_k)$ of the factors of production is limitational (weak or strong) only if it is essential.

Next suppose that a combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is essential. Then $\Phi(x) = 0$ for all $x \in D_2(\nu_1, \nu_2, \ldots, \nu_k)$. For any positive output rate u, $L(u) = \bar{E}(u) + R^n_+$ due to Proposition 4. Since $\bar{E}(u)$ is bounded (Property P.9) and closed, there exists a hyperplane which strictly separates $\bar{E}(u)$ and the closed set $\left\{x \mid x \in D_2(\nu_1, \ldots, \nu_k), \, x \in R^n_+\right\}$ of R^n [3].[†] Because all points of $L(u)$ may be expressed as $(x + y)$ where $x \in \bar{E}(u)$ and $y \geq 0$, there exists a strictly separating hyperplane

$$H : \sum_{i=1}^{k} a_i x_{\nu_i} = \alpha \, , \; \alpha > 0 \, , \; a_i > 0 \;\; \text{for} \;\; i \in \{1, 2, \ldots, k\} \, ,$$

for $L(u)$ and $\left\{x \mid x \in D_2(\nu_1, \nu_2, \ldots, \nu_k), \, x \in R^n_+\right\}$, i.e., if $x \in L(u)$ then $\sum_{i=1}^{k} a_i x_{\nu_i} > \alpha$ and if $x \in \left\{x \mid x \in D_2(\nu_1, \nu_2, \ldots, \nu_k), \, x \in R^n_+\right\}$ then $\sum_{i=1}^{k} a_i x_{\nu_i} = 0 < \alpha$. Any point

$$x_{\nu_i} = \frac{a_i \cdot \alpha}{\sum_{i=1}^{k} a_i^2} \, , \; i \in \{1, 2, \ldots, k\}$$

$$x_j \geq 0 \qquad , \; j \notin \{1, 2, \ldots, k\}$$

[†] The so-called strict separation theorem.

belon to the hyperplane H , and hence there exists a bound $\left(x^o_{\nu_1}, \ldots, x^o_{\nu_k}\right)$ $= \frac{\alpha}{\sum_1^k a_i^2}(a_1, a_2, \ldots, a_k)$ such that $\Phi(x)$ is bounded for $x \geqq 0$ and $\left(x_{\nu_1}, \ldots, x_{\nu_k}\right) \leqq \left(x^o_{\nu_1}, \ldots, x^o_{\nu_k}\right)$, because $L(v) \subset L(u)$ for $v > u$ implying $\Phi(x)$ 1 . Hence, the following proposition holds:

<u>Propo ion 9</u>: A combination $(\nu_1, \nu_2, \ldots, \nu_k)$, $1 \leqq k \leqq (n - 1)$, is weak limitational if and only if the combination is essential.

ever, essentiality by itself does not imply that a combination $(\nu_1, \ldots, \nu_k)$ is st g limitational, a fact which is easily seen from the counter example of Figur .. There, for u ranging over $[0, +\infty)$, the efficient subset $E(u) = \bar{E}(u)$ is the losed line segment $\overline{PQ}$, where $P = (0,u)$ and $Q = (u, 1 - e^{-u})$. The famil f sets so generated clearly satisfies Properties P.1, ..., P.9 for a techn gy, and it remains to show only that Property P.10 holds. For this purpo we need only consider the efficient subset $E(u)$, the points of which are n by:

$$\begin{aligned} x_1 &= \theta \cdot u \\ x_2 &= \theta(1 - e^{-u}) + (1 - \theta)u \end{aligned} \qquad \theta \in [0,1] .$$

In that $(\lambda x) \in L(\lambda u)$ for $x \in E(u)$ and $\lambda \geqq 1$, it is sufficient to sho t

$$\lambda\theta(1 - e^{-u}) + \lambda(1 - \theta)u \geqq \theta(1 - e^{-\lambda u}) + (1 - \theta)\lambda u$$

or th

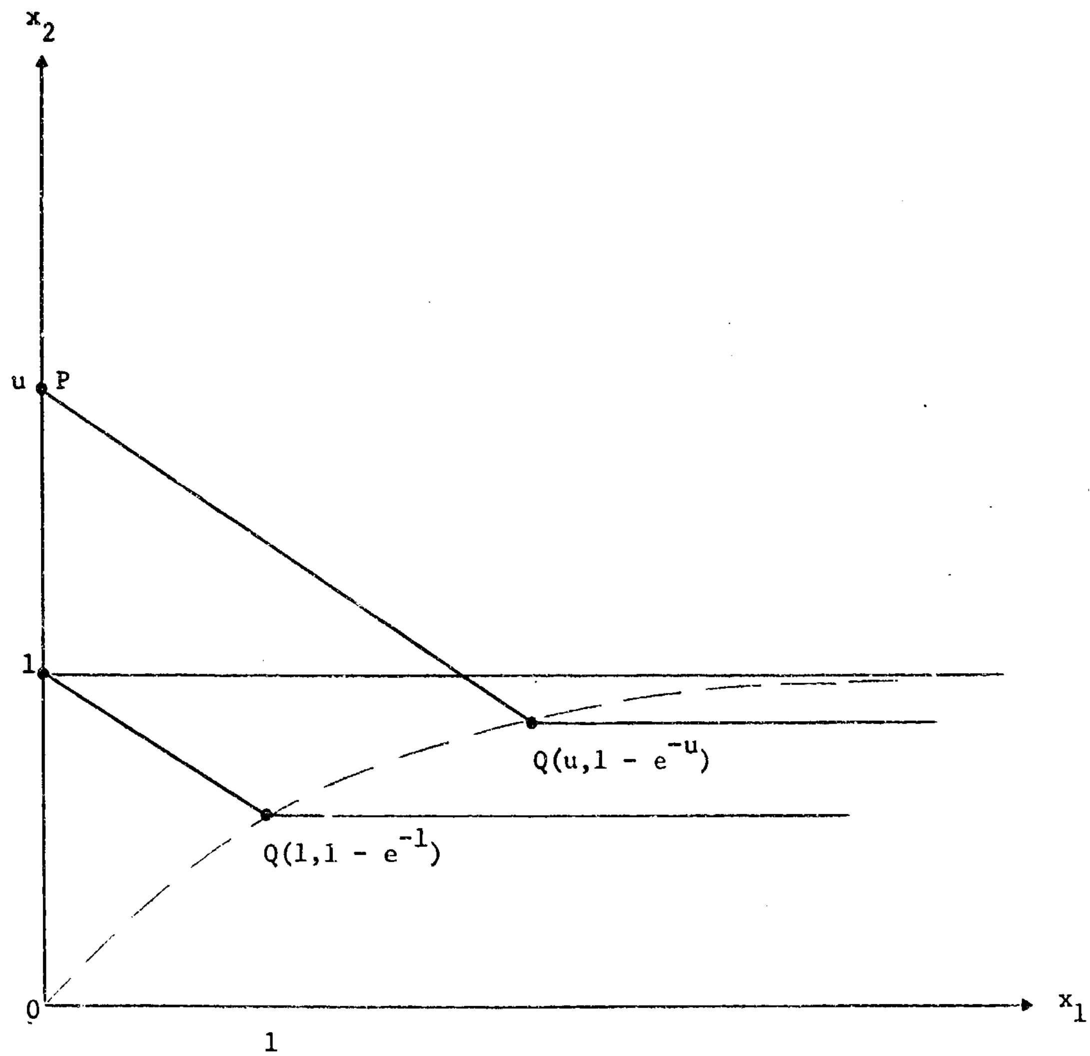

FIGURE 1: COUNTER EXAMPLE: $\overline{E(u)} = \overline{PQ}$,

$L(u) = \overline{E(u)} + \{x \mid x \geqq 0\}$

$$F(\lambda) = \lambda(1 - e^{-u}) - (1 - e^{-\lambda u}) \geqq 0$$

for all $u \geqq 0$ and $\lambda \geqq 1$. Clearly, $F(\lambda) = 0$ for $u = 0$, $\lambda \geqq 1$. Hence, take $u > 0$ and compute

$$F'(\lambda) = 1 - e^{-u} - ue^{-\lambda u}$$

$$F''(\lambda) = u^2 e^{-\lambda u} \ .$$

Since $F''(\lambda) > 0$ for $u > 0$ and $\lambda \geqq 1$, $F(\lambda)$ is strictly convex in λ for arbitrary $u > 0$. Now, $F(1) = 0$ and

$$F'(1) = 1 - \frac{1 + u}{e^u} > 0$$

for arbitrary $u > 0$. Hence, $F(\lambda) \geqq 0$ for $\lambda \geqq 1$ and $u \geqq 0$.

Similarly, in order that $x \in \frac{1}{\lambda} L(u)$ for $x \in E\left(\frac{u}{\lambda}\right)$ and $\lambda \geqq 1$, it is sufficient to show that

$$G(\lambda) = \lambda\left(1 - e^{-\frac{u}{\lambda}}\right) - (1 - e^{-u}) \geqq 0$$

for all $u \geqq 0$ and $\lambda \geqq 1$. Since $G(\lambda) = 0$ for $u = 0$ and $\lambda \geqq 1$, take $u > 0$ arbitrarily and

$$G'(\lambda) = 1 - e^{-\frac{u}{\lambda}} - \frac{u}{\lambda} e^{-\frac{u}{\lambda}}$$

$$= 1 - \frac{1 + \frac{u}{\lambda}}{e^{\frac{u}{\lambda}}} > 0$$

for all $\lambda \geqq 1$. Since $G(1) = 0$ for $u \geqq 0$ and $G(\lambda)$ is strictly increasing in λ for $\lambda \geqq 1$ and $u > 0$, it follows that $G(\lambda) \geqq 0$ for all $\lambda \geqq 1$ and $u \geqq 0$.

Hence, the example of Figure 1 satisfies all the properties required of a technology. In this example, the factor of production with input denoted by x_1 is not essential, while the factor with input x_2 is essential because any input vector of the form $(x_1,0)$ does not belong to any input set $L(u)$ for $u > 0$ and $\Phi(x_1,0) = 0$ for all $x_1 \geqq 0$. For any positive bound x_2^o such that $x_2^o < 1$, $\Phi(x)$ is bounded for $x_1 \geqq 0$ and $0 \leqq x_2 \leqq x_2^o$, while $\Phi(x)$ is unbounded for $x_2^o \geqq 1$, $x_1 \geqq 0$ and $0 \leqq x_2 \leqq x_2^o$. Thus, we have a counter example against the essentiality of a combination $(\nu_1,\nu_2, \ldots, \nu_k)$ implying that the combination is strong limitational.

Recall, however, that, if the combination $(\nu_1,\nu_2, \ldots, \nu_k)$, $1 \leqq k \leqq (n - 1)$, is essential it has been shown for arbitrary $u > 0$ that there exists a separating hyperplane

$$\sum_{i=1}^{k} a_i x_{\nu_i} = \alpha \ , \ \alpha > 0 \ , \ a_i > 0 \quad \text{for} \quad i \in \{1,2, \ldots, k\} \ ,$$

where in general α may depend upon the output rate u . Let

$$S(u) = \operatorname*{Sup}_{x} \left\{ \sum_{i=1}^{k} a_i x_{\nu_i} \mid x \notin L(u) \right\} ,$$

and consider an increasing sequence of output rates $\{u_n\} \to +\infty$. If the corresponding sequence $\{S(u_n)\}$ is unbounded, $\Phi(x)$ is bounded for any positive bound $\left(x_{\nu_1}^o, \ldots, x_{\nu_k}^o\right)$ on an essential combination of the factors of production. Contrariwise, if the sequence $\{S(u_n)\}$ is bounded, it converges to a limit S_o and the hyperplane

$$\sum_{i=1}^{k} a_i x_{\nu_i} = S_o \ , \ a_i > 0 \quad \text{for} \quad i \in \{1,2, \ldots, k\}$$

intersects all production sets $L(u)$, $u \in [0,+\infty)$, whence, for any bound

$$x_{\nu_i} \leqq x^o_{\nu_i} = \frac{a_i}{\sum_1^k a_i^2} \cdot A_o \qquad i \in \{1,2, \ldots, k\}$$

with $A \geqq S_o$, $x_{\nu_j} \geqq 0$ for $j \notin \{1,2, \ldots, k\}$, $\Phi(x)$ is unbounded and the combination $(\nu_1,\nu_2, \ldots, \nu_k)$ is not strong limitational. Thus, the following proposition holds:

Proposition 10: A combination $(\nu_1,\nu_2, \ldots, \nu_k)$ is strong limitational if and only if the combination is essential and

$$\lim_{n\to\infty} \operatorname{Sup} \left[\operatorname{Sup}_x \left\{ \sum_1^k a_i x_{\nu_k} \;\middle|\; x \in L(u_n) \right\} \right] = +\infty$$

for $a_i > 0$, $i \in \{1,2, \ldots, k\}$ and $\{u_n\} \to +\infty$.

If the second condition of Proposition 10 does not hold, it is implied, for some bounded positive inputs for an essential combination of the factors of production, that output is unbounded for unrestricted inputs of the other factors.

5. THE PRODUCTION STRUCTURE OF A TECHNOLOGY WITH PARTIALLY BOUNDED INPUTS

The considerations of the previous section lead one to consider the production possibility sets of a technology when the inputs of some (but not all) of the factors of production are limited by some positive bounds.

Let $(\nu_1, \nu_2, \ldots, \nu_k)$, $1 \leqq k \leqq (n - 1)$, be a combination of the factors of production and suppose that the input of the factors of this combination are bounded by

$$0 \leqq x_{\nu_i} \leqq x^o_{\nu_i} \ , \ \nu^o_i > 0 \ , \ i \in \{1,2, \ldots, k\} \ .$$

Let

$$x_k = \left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right)$$

$$y_k = \left(x_{\nu_{k+1}}, x_{\nu_{k+2}}, \ldots, x_{\nu_n}\right)$$

denote subvectors of the input vector x , and take $x = (x_k, y_k)$ since the order is not important. Define

$$D^o = \left\{(x_k, y_k) \mid x_k \leqq x^o_k \ , \ x \geq 0\right\}$$

$$L^o(u) = L(u) \cap D^o \ , \ u \in [0,+\infty) \ .$$

The sets $L^o(u)$ are the production possibility sets for the limited operation of the technology when $0 \leqq x_k \leqq x^o_k$.

It is rather straightforward to verify that the sets $L^o(u)$, $u \in [0,+\infty)$ satisfy analogues of the Properties P.4, ..., P.8 for the production sets of an unrestricted technology, with P.1 and P.2 replaced by

P.1° $L^{\circ}(0) = D^{\circ}$.

P.2° If $x \in D^{\circ}$, $x' \in D^{\circ}$, $x' \geqq x$ and $x \in L^{\circ}(u)$, then $x' \in L^{\circ}(u)$.

Regarding Property P.3, suppose first that the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is nonessential. Then Propositions 5 and 6 imply that if $y_k > 0$ there exists a scalar λ_u for any $u > 0$ such that $\Phi(0, \lambda_u \cdot y_k) > 0$. Then by Properties A.1, A.3 and A.9, it follows, for any x_k for which $0 \leqq x_k \leqq x_k^{\circ}$, that

$$\begin{aligned} \Phi(x_k, \lambda_u \cdot y_k) &= \Phi((x_k, 0) + (0, \lambda_u \cdot y_k)) \\ &\geqq \Phi(0, \lambda_u \cdot y_k) \geqq u \ , \end{aligned}$$

or for any $y_k > 0$ there exists a scalar λ_u (depending also perhaps on y_k) such that $(x_k, \lambda_u \cdot y_k) \in L^{\circ}(u)$. Thus, the following property holds:

P.3°(a) If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is nonessential:

(i) $L^{\circ}(u)$ is nonempty for all $u \in [0, +\infty)$.

(ii) For $(x_k, y_k) \in D^{\circ}$ and $y_k > 0$, the ray $\{(x_k, \lambda y_k) \mid \lambda \geqq 0\}$ emanating from the point $(x_k, 0)$ intersects all sets $L^{\circ}(u)$ for $u \in [0, +\infty)$.

Assume now that the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is essential. Two situations arise. Either the combination is weak limitational or it is strong limitational. If it is strong limitational, $\Phi(x_k, y_k)$ is bounded for $(x_k, y_k) \in D^{\circ}$ and not all sets $L^{\circ}(u)$ are nonempty. If it is weak limitational, it may happen that $\Phi(x_k, y_k)$ is unbounded for $(x_k, y_k) \in D^{\circ}$. Then for $y_k > 0$, $\Phi\left(x_k^{\circ}, \lambda y_k\right) \to +\infty$ as $\lambda \to +\infty$, and for any $u > 0$ there exists a scalar λ_u such that $\left(x_k^{\circ}, \lambda_u \cdot y_k\right) \in L^{\circ}(u)$, and the sets $L^{\circ}(u)$ are nonempty for $u \in [0, +\infty)$. Hence, Property P.3 takes the following second form when the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is essential:

P.3°(b) If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is weak limitational† and $\phi(x_k, y_k)$ is unbounded for $(x_k, y_k) \in D^o$:

(i) $L^o(u)$ is nonempty for all $u \in [0, +\infty)$.

(ii) For $y_k > 0$, the ray $\left\{\left(x_k^o, \lambda y_k\right) \mid \lambda \geqq 0\right\}$ emanating from the point $\left(x_k^o, 0\right)$ intersects all sets $L^o(u)$ for $u \in [0, +\infty)$.

It remains to consider Properties P.9 and P.10. The efficient subset $E^o(u)$ of a nonempty limited production possibility set $L^o(u)$ is defined by

$$E^o(u) = \left\{(x_k, y_k) \mid (x_k, y_k) \in L^o(u) \text{ , } (x_k', y_k') \notin L^o(u) \text{ if } (x_k', y_k') \leq (x_k, y_k)\right\} .$$

An efficient subset $E^o(u)$ is nonempty if $L^o(u)$ is nonempty (see [11], Section 4, or [12], Section 2.8, for proof) and $E^o(u) \subset E(u)$. Thus, Property P.9 holds for the limited production sets $L^o(u)$.

It is easy to verify that Property P.10 does not hold for the sets $L^o(u)$. However, a useful modification does hold. Consider the situation where the bound $x_k^o = 0$ and the combination $(\nu_1, \ldots, \nu_k)$ is nonessential. The resulting sets $L^o(u)$, $u \in [0, +\infty)$, are nonempty (see Property P.3°(a)) and they represent a technology with the factors $(\nu_1, \nu_2, \ldots, \nu_k)$ omitted, i.e., one with more limited alternatives but nevertheless a technology. Consequently, for these sets, if $(0, \bar{y}_k) \in L(u)$ then $\lambda(0, \bar{y}_k) = (0, \lambda\bar{y}_k) \in L(\lambda u)$ for $\lambda \geqq 1$. Now return to the case where $x_k^o > 0$ and suppose $(x_k, y_k) \in L^o(u)$ and $y_k > 0$.†† For a sufficiently large magnification of y_k , i.e., $\theta_u \cdot y_k$ such that $(0, \theta_u \cdot y_k) \in L(u)$,

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†Note that essential and weak limitational are synonymous (see Proposition 9), not necessarily implying that the combination is strong limitational.

††If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is maximally nonessential, i.e., $(\nu_1, \ldots, \nu_k, \nu_j)$ is essential for all $j \in \{k+1, \ldots, n\}$, then $(x_k, y_k) \in L^o(u)$ implies $y_k > 0$.

$(x_k,\theta_u \cdot y_k) \in L^o(u)$ implies for $\lambda \geqq 1$ that $(x_k,\lambda\theta_u \cdot y_k) \in L^o(\lambda u)$, because $(0,\lambda\theta_u \cdot y_k) \in L(u)$ and $(x_k,\lambda\theta_u \cdot y_k) \in L^o(\lambda u)$ due to Property P.2^o. Thus, the following property holds for the sets $L^o(u)$:

P.10^o(a) If $(\nu_1, \ldots, \nu_k)$ is nonessential and $y_k > 0$, or the combination $(\nu_1,\nu_2, \ldots, \nu_k)$ is maximally nonessential, then, for a sufficiently large magnification $\theta_u \cdot y_k$ such that $(0,\theta_u \cdot y_k) \in L(u)$, $(x_k,\theta_u \cdot y_k) \in L^o(u)$ implies $(x_k,\lambda\theta_u \cdot y_k) \in L^o(\lambda u)$ for $\lambda \geqq 1$.

Turn now to the situation where the combination $(\nu_1,\nu_2, \ldots, \nu_k)$ is essential. Two situations arise. If the combination is strong limitational, then for any bound $x_k^o > 0$, $\Phi(x_k,y_k)$ is bounded for $0 \leqq x_k \leqq x_k^o$ and $y_k \geqq 0$. Denote this bound by

$$u\left(x_k^o\right) = \sup_{\lambda} \left\{\Phi\left(x_k^o,\lambda y_k\right) \mid y_k > 0 \ , \ \lambda \geqq 0\right\} .$$

If $L^o(u)$ is nonempty, $(x_k,y_k) \in L^o(u)$ and $\lambda > \dfrac{u\left(x_k^o\right)}{u}$, it follows that

$$\Phi(x_k,\lambda y_k) < \lambda u \leqq \lambda\Phi(x_k,y_k) .$$

If the combination $(\nu_1,\nu_2, \ldots, \nu_k)$ is only weak limitational, there exists a bound $\bar{x}_k^o > 0$ such that $\Phi(x_k,y_k)$ is bounded for $0 \leqq x_k \leqq \bar{x}_k^o$, $y_k \geqq 0$; and, if $L^o(u)$ is non-empty, $(x_k,y_k) \in L^o(u)$ and $\lambda > \dfrac{u\left(\bar{x}_k^o\right)}{u}$, $\Phi(x_k,\lambda y_k) < \lambda\Phi(x_k,y_k)$. But, in this latter situation, if $x_k^o > \bar{x}_k^o$, $\Phi(x_k,y_k)$ is unbounded for $0 \leqq x_k \leqq x_k^o$, $y_k \geqq 0$, and nothing can be said without assumptions on the fine structure of the technology $T : L(u)$, $u \in [0,+\infty)$.

In summary, the following proposition holds regarding the production possibility sets $L^o(u)$ of a limited technology:

<u>Proposition 11</u>: If a subvector $\left(x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}\right) = x_k$ of a technology $T : L(u)$ is constrained to $0 \leqq x_k \leqq x_k^o$, $x_k^o > 0$, with $y_k = \left(x_{\nu_{k+1}}, \ldots, x_{\nu_n}\right) \geqq 0$, the production sets $L^o(u) = L(u) \cap D^o$, $D^o = \left\{(x_k, y_k) \mid x_k \leqq x_k^o , y_k \geqq 0\right\}$, satisfy (let $x = (x_k, y_k)$)

P.1° $\quad L^o(0) = D^o$.

P.2° $\quad$ If $x \in D^o$, $x' \in D^o$, $x' \geqq x$ and $x \in L^o(u)$, $x' \in L^o(u)$.

P.3°(a) $\quad$ If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is nonessential:

(i) $L^o(u)$ is nonempty for all $u \in [0,+\infty)$.

(ii) For $(x_k, y_k) \in D^o$ and $y_k > 0$, the ray $\{(x_k, \lambda y_k) \mid \lambda \geqq 0\}$ emanating from the point $(x_k, 0)$ intersects all sets $L^o(u)$ for $u \in [0,+\infty)$.

(b) $\quad$ If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is weak limitational and [illegible](x_k, y_k) is unbounded for $(x_k, y_k) \in D^o$:

(i) $L^o(u)$ is nonempty for all $u \in [0,+\infty)$.

(ii) For $y_k > 0$, the ray $\left\{\left(x_k^o, \lambda y_k\right) \mid \lambda \geqq 0\right\}$ emanating from the point $\left(x_k^o, 0\right)$ intersects all sets $L^o(u)$ for $u \in [0,+\infty)$.

P.4° $\quad u_2 \geqq u_1 \geqq 0$ implies $L^o(u_2) \subset L^o(u_1)$.

P.5° $\quad \bigcap_{u \in [0,+\infty)} L^o(u)$ is empty.

P.6° $\quad \bigcap_{0 \leqq u < u_o} L^o(u) = L^o(u_o)$ for $u_o > 0$.

P.7° $L^{O}(u)$ is closed for all $u \in [0,+\infty)$.

P.8° $L^{O}(u)$ is convex for all $u \in [0,+\infty)$.

P.9° $E^{O}(u)$ is bounded for all $u \in [0,+\infty)$.

P.10°(a) If $(\nu_1, \ldots, \nu_k)$ is nonessential and $y_k > 0$, or the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is maximally nonessential, then for a sufficiently large magnification $\theta_u \cdot y_k$ such that $(0, \theta_u \cdot y_k) \in L(u)$, $(x_k, \theta_u \cdot y_k) \in L^{O}(u)$ implies $(x_k, \lambda\theta_u \cdot y_k) \in L^{O}(\lambda u)$ for $\lambda \geqq 1$.

(b) If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is only weak limitational: then $x_k > 0$,

$\operatorname*{Sup}_{\lambda} \{\Phi(x_k, \lambda y_k) \mid y_k > 0 , \lambda \geqq 0\} = u(x_k)$ is finite,

$L^{O}(u)$ is nonempty, $(x_k, y_k) \in L^{O}(u)$ and $\lambda > \dfrac{u(x_k)}{u}$,

imply $\Phi(x_k, \lambda y_k) < \lambda\Phi(x_k, y_k)$.

(c) If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is strong limitational: then for any subvector $x_k > 0$,

$\operatorname*{Sup}_{\lambda} \{\Phi(x_k, \lambda y_k) \mid y_k > 0 , \lambda \geqq 0\} = u(x_k)$ is finite,

and $L^{O}(u)$ nonempty, $(x_k, y_k) \in L^{O}(u)$, $\lambda > \dfrac{u(x_k)}{u}$

imply $\Phi(x_k, \lambda y_k) < \lambda\Phi(x_k, y_k)$.

6. LAWS OF DIMINISHING RETURNS

The properties assumed for the production possibility sets $L(u)$ of a technology only macroscopically characterize the production structure. Nothing has been assumed about the fine structure.

Postulate for Essential Factors: There exists at least one essential combination of the factors of production for a technology, exclusive of $(\nu_1, \nu_2, \ldots, \nu_n)$.

If all combinations of k factors are nonessential for $1 \leqq k \leqq n - 1$, unbounded output can be obtained for all bounds on the inputs of any of these combinations and diminishing returns is not implied in any sense unless assumptions are made on the fine structure which are contrived to obtain the result.

Definition: An essential combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is minimal if the combinations obtained by deleting any one of the factors x_{ν_i} , $i \in \{1, 2, \ldots, k\}$, are nonessential.

Let $h_k = \left\{h_{\nu_{k+1}}, h_{\nu_{k+2}}, \ldots, h_{\nu_n}\right\} > 0$ denote a positive increment for the vector y_k .

Weak Law of Diminishing Product Increments: For every combination $(\nu_1, \nu_2, \ldots, \nu_k)$, $k_o \leqq k \leqq (n - 1)$ related to a minimal essential combination $\left(\nu_1, \nu_2, \ldots, \nu_{k_o}\right)$ of the factors of production, there exists a positive bound x_k^o such that $\Phi(x_k, y_k)$ is bounded for $0 \leqq x_k \leqq x_k^o$, $y_k \geqq 0$ and

either (a) for every input vector $(\bar{x}_k,\bar{y}_k)$, $0 \leqq \bar{x}_k \leqq x_k^o$ such that

$\Phi(\bar{x}_k,\bar{y}_k) > 0$, there exists a subvector $\hat{y}_k$ such that

$\Phi(\bar{x}_k,y_k + h_k) - \Phi(\bar{x}_k,y_k) < \Phi(\bar{x}_k,\bar{y}_k + h_k) - \Phi(\bar{x}_k,\bar{y}_k)$ for

every $y_k > \hat{y}_k$ and h_k such that $\Phi(\bar{x}_k,\bar{y}_k + h_k) > \Phi(\bar{x}_k,\bar{y}_k)$.

or (b) $\Phi(\bar{x}_k,y_k) = 0$ for all $y_k > 0$ for every input vector $\bar{x}_k$

such that $\Phi(\bar{x}_k,y_k)$ is bounded for $y_k \geqq 0$.

The "or part" of this weak law is required, because, although the essentiality of the combinations $(\nu_1,\nu_2, \ldots, \nu_k)$, $k_o \leqq k \leqq (n - 1)$ implies the existence of a positive bound x_k^o such that $\Phi(x_k,y_k)$ is bounded for $0 \leqq x_k \leqq x_k^o$, $y_k \geqq 0$, this bound may be zero. See the example of Figure 2. The family of production input sets $L(u)$ illustrated satisfies the Properties P.1, ..., P.10. If $0 < x_2^o < 1$, $\Phi(x_1,x_2) = 0$ for all $0 \leqq x_2 \leqq x_2^o$, $x_1 \geqq 0$ and $\Phi(x_1,x_2)$ is bounded for $x_1 \geqq 0$ only if $0 < x_2^o < 1$.

For the "either part," the weak law of diminishing product increments has the form which Menger [6] refers to as an "assertion intersecting" a "proposition of diminishing product increments," the latter implying that the production function is a strictly concave function of y_k for the given $\bar{x}_k$ provided y_k is sufficiently large (i.e., $y_k > \hat{y}_k$) , while the "intersecting assertion" implies merely that if $y_k > \hat{y}_k$ the product increment associated with $h_k > 0$ is smaller when applied to y_k than when applied to $\bar{y}_k$. Note that $\hat{y}_k$ depends generally upon $\bar{x}_k$, $\bar{y}_k$ and h_k .

A property of strict concavity (in y_k) of the function $\Phi(\bar{x}_k,y_k)$ for $y_k > \hat{y}_k$ cannot be deduced from the properties used to define a technology. It is an over simplified statement of diminishing returns. The weak law of diminishing product increments is not an empty statement, since at least one essential combination of factors is postulated for a technology.

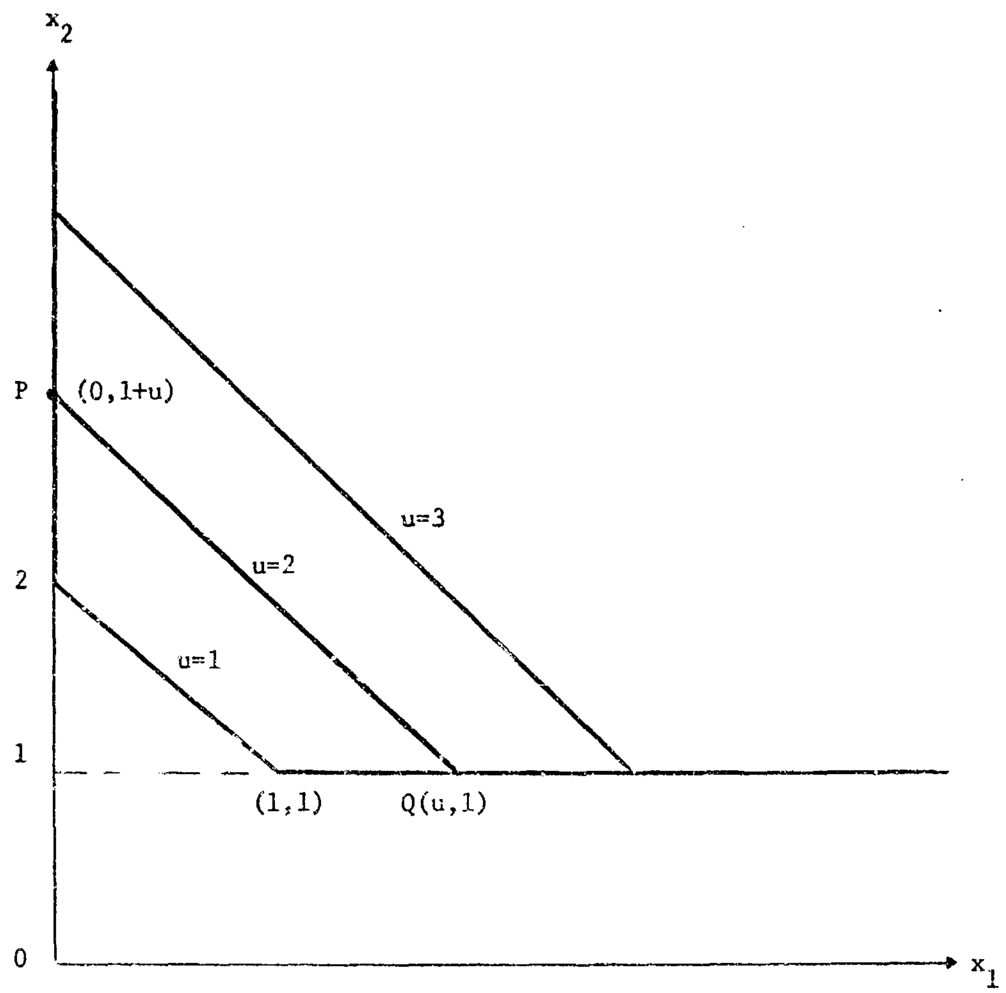

FIGURE 2: A FAMILY OF PRODUCTION INPUT SETS: $E(u) = \overline{PQ}$, $P = (0,1 + u)$, $Q = (u,1)$ FOR $u > 0$. $L(0) = R_+^2$

The existence of the bound x_k^o such that $\Phi(\bar{x}_k, y_k)$ is bounded for $0 \leqq \bar{x}_k \leqq x_k^o$, $y_k \geqq 0$ follows from Proposition 9. For the proof of statement (a), let $\Phi(\bar{x}_k, y_k)$ be bounded for $y_k \geqq 0$ and $\Phi(\bar{x}_k, \bar{y}_k) > 0$ for some $\bar{y}_k > 0$. Let h_k be a positive increment h_k such that $\Phi(\bar{x}_k, \bar{y}_k + h_k) > \Phi(\bar{x}_k, \bar{y}_k)$. Let

$$y_k^{(1)} < y_k^{(1)} + h_k < y_k^{(2)} < y_k^{(2)} + h_k < \dots < y_k^{(n)} < y_k^{(n)} + h_k < \dots$$

be a strictly increasing infinite sequence of subvectors. Since the corresponding infinite sequence

$$\Phi\left(\bar{x}_k, y_k^{(1)}\right) \leqq \Phi\left(\bar{x}_k, y_k^{(1)} + h_k\right) \leqq \dots \leqq \Phi\left(\bar{x}_k, y_k^{(n)}\right) \leqq \Phi\left(\bar{x}_k, y_k^{(n)} + h_k\right) \leqq \dots$$

is nondecreasing (Property A.3) and bounded, it converges to a limit. Accordingly this sequence of output rates is a Cauchy sequence [8], and there exists for any positive ε an integer $N(\varepsilon)$ such that for $n > N(\varepsilon)$, $\Phi\left(\bar{x}_k, y_k^{(n)} + h_k\right) - \Phi\left(\bar{x}_k, y_k^{(n)}\right) < \varepsilon$. Hence, there is a subvector $\hat{y}_k$, depending upon the difference $d = \Phi(\bar{x}_k, \bar{y}_k + h_k) - \Phi(\bar{x}_k, \bar{y}_k)$, such that for $y_k > \hat{y}_k$,

$$\Phi(\bar{x}_k, y_k + h_k) - \Phi(\bar{x}_k, y_k) < \Phi(\bar{x}_k, \bar{y}_k + h_k) - \Phi(\bar{x}_k, \bar{y}_k) \ .$$

Statement (b) is merely a statement of the possibility illustrated in Figure 2. There, for any product increment $h_k > 0$ the product differences are all equal to zero.

Weak Law of Diminishing Average Product: If the combination $(\nu_1, \nu_2, \dots, \nu_k)$ is weak limitational there exists a bound $x_k^o > 0$ such that $\Phi(x_k, y_k)$ is bounded for $0 \leqq x_k \leqq x_k^o$, $y_k \geqq 0$. Then for every x_k such that $0 \leqq x_k \leqq x_k^o$, $\sup_{\lambda} \{\Phi(x_k, \lambda y_k) \mid y_k > 0 \ , \ \lambda \geqq 0\} = u(x_k)$ is finite and, if $u > 0$, $L^o(u)$ is nonempty, $(x_k, y_k) \in L^o(u)$ and $\lambda > \dfrac{u(x_k)}{u}$, then $\Phi(x_k, \lambda y_k) < \lambda\, \Phi(x_k, y_k)$.

This law is merely a restatement of Property P.10° (b) for the production possibility sets $L^{O}(u)$ of a limited technology. It has the form described by Menger [6] as an "assertion intersecting" a "proposition of diminishing average product," the latter implying that beyond some input $\hat{y}_k$, i.e., for $\lambda y_k > \hat{y}_k$ the average return $\Phi(x_k, \lambda y_k)/\lambda$ is strictly decreasing, while the "intersecting assertion" implies merely that for any $y_k > 0$ there exists a value $\frac{u(x_k)}{u}$ such that for $\lambda > \frac{u(x_k)}{u}$ the average return is less than the positive output associated with (x_k, y_k) , when x_k does not exceed the bound x_k^o . The existence of the bound x_k^o follows from Proposition 9. Strictly decreasing average returns for $\lambda y_k > \hat{y}_k$ is a property of the fine structure of the technology and cannot be deduced without assumptions contrived to obtain this result.

<u>Strong Law of Diminishing Product Increments</u>: For every combination $(\nu_1, \nu_2, \ldots, \nu_k)$, $k_o \leqq k \leqq (n - 1)$ related to a minimal essential combination $\nu_1, \nu_2, \ldots, \nu_{k_o}$ of the factors of production which is strong limitational,

$$\Phi(\bar{x}_k, y_k + h_k) - \Phi(\bar{x}_k, y_k) < \Phi(\bar{x}_k, \bar{y}_k + h_k) - \Phi(\bar{x}_k, \bar{y}_k)$$

if $\Phi(\bar{x}_k, \bar{y}_k) > 0$, $\Phi(\bar{x}_k, \bar{y}_k + h_k) > \Phi(\bar{x}_k, \bar{y}_k)$ and $y_k > \hat{y}_k$ where $\hat{y}_k$ depends upon $\bar{x}_k$, $\bar{y}_k$ and h_k .

In this strong law no restriction is put upon the vector x_k other than $\Phi(\bar{x}_k, \bar{y}_k) > 0$, i.e., it is not bounded, because for any fixed input $x_k > 0$ the output $\Phi(x_k, y_k)$ is bounded for $y_k \geqq 0$, and the proof follows exactly that given for the corresponding weak law. The statement (b) is omitted because due to Property A.4 and the property of strong limitational there exists a vector $(\bar{x}_k, \bar{y}_k)$ such that $\Phi(\bar{x}_k, \bar{y}_k) > 0$ and $\Phi(\bar{x}_k, y_k) \neq 0$ for $y_k \geqq 0$.

<u>Strong Law of Diminishing Average Product</u>: If the combination $(\nu_1, \nu_2, \ldots, \nu_k)$ is strong limitational, then, for every $x_k > 0$, $\operatorname{Sup}_\lambda \{\Phi(x_k, \lambda y_k) \mid y_k > 0, \lambda \geqq 0\} = u(x_k)$ is finite and, if $u > 0$, $L^0(u)$ is nonempty, $(x_k, y_k) \in L^0(u)$ and $\lambda > \frac{u(x_k)}{u}$. Then $\Phi(x_k, \lambda y_k) < \lambda\Phi(x_k, y_k)$.

For this strong law, no restriction is put on x_k because $\Phi(x_k, y_k)$ is bounded for $y_k \geqq 0$ and x_k fixed, since the combination is strong limitational.

The foregoing laws are precise laws of diminishing returns for any technology $T : L(u)$, $u \in [0, +\infty)$, and provable for such structures without assumptions on the fine structure of T. Nothing is said about any particular physical production system. It is presumed, however, that the ideal structure T describes macroscopically all actual production systems. Only in this sense does a law of diminishing returns have meaning. If an actual physical production system can be found which violates the laws, excepting situations where the output u for an input vector x does not correspond to $\Phi(x) = \operatorname{Max} \{u \mid x \in L(u)\}$, i.e., inefficient systems, then the properties defining the technology T must be modified in some way to encompass this critical observation and new forms of the laws sought which are not contradicted.

7. THE COBB-DOUGLAS AND CES PRODUCTION FUNCTIONS AND RESTRICTED LAWS OF DIMINISHING RETURNS

It is useful to look at some functions which are commonly used in econometric studies, i.e., the Cobb-Douglas and CES functions. The Cobb-Douglas function [4] may be represented by

$$\Phi(x) = \Phi_o \prod_{1}^{n} \left(\frac{x_i}{x_i^o}\right)^{a_i} ,$$

with $a_i > 0$, $x_i^o > 0$ $(i = 1, \ldots, n)$ and $\sum_1^n a_i = 1$. The quantities x_i^o are some positive inputs at a reference point of the set R_+^n, taken to give an expression which is independent of the diverse physical units of the factors of production. This function does not satisfy Property P.9 for the implied input sets $L(u)$ (or Property A.8), and hence it is not a valid production function over the entire domain R_+^n of the input vectors x. It has the further restrictive property that each factor of production is essential, that is, no factor may be completely substituted for another.

Similarly, the CES f ction [2] fails to be a valid production function over the entire domain R_+^n of the input vectors foı significant parameter values. This function, presented for two aggregate factors of production (capital and labor) by the expression

$$\Phi(x) = \left[a_1 x_1^{-\beta} + a_2 x_2^{-\beta}\right]^{-\frac{1}{\beta}} ,$$

with $a_1 > 0$, $a_2 > 0$, $\beta > -1$, $\beta \neq 1$, is offered as a "new class of production functions," but the efficient subsets of the implied production possibility sets are not bounded (i.e., P.9 fails to hold) when $\beta > 0$, the case described by the authors as the most interesting case, because then diminishing returns holds. Also,

when the expression is written for n factors of production and $\beta > 0$, it has the restrictive property that each factor of production is essential.

For both functions, output is unbounded (when diminishing returns holds) if a positive bound is put on any factor of production and the others are allowed to indefinitely increase, that is the factors are individually essential but not even weak limitational!

From three premises concerning the production function: (a) increase in output for an appropriate increase of an input x_j, (b) positive homogeneity of degree one, and (c) homogeneity in $(n - 1)$ factors for fixed input of the remaining factor, all of which are satisfied by the Cobb-Douglas function (*but not boundedness of the efficient subsets*), Eichhorn [5] has deduced that the production function satisfies an over the whole range strictly decreasing product increments.

For the class of production structures which (in addition to P.1, ..., P.10) are positively homogeneous of degree one, the following proposition holds.

<u>Proposition 12</u>: If the technology satisfies $L(\lambda u) = \lambda L(u)$ for all $\lambda > 0$, the production function $\Phi(x)$ is positively homogeneous of degree one and a concave and continuous function of x on R^n_+.

For $\lambda > 0$,

$$\begin{aligned}\Phi(\lambda x) &= \operatorname{Max}\{u \mid (\lambda x) \in L(u),\ u \geq 0\}\\ &= \operatorname{Max}\left\{u \mid x \in L\left(\frac{u}{\lambda}\right),\ u \geq 0\right\}\\ &= \lambda \operatorname{Max}\left\{\frac{u}{\lambda} \mid x \in L\left(\frac{u}{\lambda}\right),\ \frac{u}{\lambda} \geq 0\right\}\\ &= \lambda\Phi(x)\ .\end{aligned}$$

From the homogeneity of $\Phi(x)$ it follows that the production function is a concave and continuous function of x for all $x \in R_+^n$. See [11], p. 226 or [12], Section 2.6 for a proof of this statement.

Thus, a restricted law of nonincreasing product increments holds:

<u>Restricted Law of Nonincreasing Returns</u>: If the production structure is positively homogeneous of degree one, then for any input vector x^o such that $\Phi(x^o) > 0$ and arbitrary increment $h = (h_1, h_2, \ldots, h_n)$ subject to $h_{\nu_i} = 0$ for $i \in \{1,2, \ldots, k\}$ and $h_{\nu_j} > 0$ for $j \notin \{1,2, \ldots, k\}$, $1 \leqq k \leqq (n-1)$, such that $\Phi(x^o + (N-1)h) - \Phi(x^o + (N-2)h) > 0$ for $N \geqq 2$, then

$$\Phi(x^o + N\cdot h) - \Phi(x^o + (N-1)\cdot h) \leqq \Phi(x^o + (N-1)\cdot h) - \Phi(x^o + (N-2)\cdot h)$$

for all integers $N \geqq 2$.

For any $\theta \in [0,1]$, consider $\bar{x} = x^o + (N-2)h$, $N \geqq 2$, and

$$\theta\cdot\bar{x} + (1-\theta)(\bar{x} + 2h) = \bar{x} + 2(1-\theta)h \ .$$

Then, from the concavity of $\Phi(x)$,

$$\Phi(\theta\cdot\bar{x} + (1-\theta)(\bar{x} + 2h) \geqq \theta\Phi(\bar{x}) + (1-\theta)\Phi(\bar{x} + 2h) \ ,$$

and for $\theta = \frac{1}{2}$,

$$2\Phi(\bar{x} + h) \geqq \Phi(\bar{x}) + \Phi(\bar{x} + 2h) \ ,$$

whence

$$\Phi(\bar{x} + 2h) - \Phi(\bar{x} + h) \leqq \Phi(\bar{x} + h) - \Phi(\bar{x})$$

and

$$\Phi(x^o + N\cdot h) - \Phi(x^o + (N-1)h) \leqq \Phi(x^o + (N-1)h) - \Phi(x^o + (N-2)h)$$

for all $N \geqq 2$.

Further, if the Property P.9 for the input sets $L(u)$ is deleted for $u > 0$, and the sets $L(u)$ are assumed to be strictly convex for $u > 0$ (which is the case for the Cobb-Douglas and CES functions), the following restricted law for strictly decreasing returns holds:

Restricted Law of Decreasing Returns: If the production structure is positively homogeneous of degree one, Property P.9 is deleted for $u > 0$ and the input sets $L(u)$ are strictly convex for $u > 0$, then for any input vector x^o such that $\Phi(x^o) > 0$ and arbitrary increment $h = (h_1, h_2, \ldots, h_n)$ with $h_{\nu_i} = 0$ for $i \in \{1,2,\ldots,k\}$ and $h_{\nu_j} > 0$ for $j \notin \{1,2,\ldots,k\}$, for $1 \leqq k \leqq (n-1)$, such that $\Phi(x^o + (N-1)h) - \Phi(x^o + (N-2)h) > 0$ for $N \geqq 2$, then

$$\Phi(x^o + Nh) - \Phi(x^o + (N-1)h) < \Phi(x^o + (N-1)h) - \Phi(x^o + (N-2)h)$$

for $N \geqq 2$.

The strict convexity of the input sets implies that if $\Phi(x) > 0$, $\Phi(y) > 0$ then $\Phi(x+y) > \Phi(x) + \Phi(y)$, i.e., for input vectors x and y yielding positive output the production function is strictly super additive. To see this, note that for any $u > 0$

$$\Phi\left(\frac{u}{\Phi(x)}\cdot x\right) = \Phi\left(\frac{u}{\Phi(y)}\cdot y\right) = u$$

due to the homogeneity of $\Phi(\cdot)$. Hence, the input vectors

$$\frac{u}{\Phi(x)}\cdot x \; , \; \frac{u}{\Phi(y)}\cdot y$$

belong to the boundary of $L(u)$, because if not, say for the first, then

$$\theta \frac{u}{\Phi(x)}\cdot x \in \text{Boundary of } L(u)$$

for $\theta < 1$ and $\Phi\left(\theta\cdot\frac{u}{\Phi(x)}\cdot x\right) = \theta u \geqq u$ for $\theta < 1$, a contradiction. Consequently, for any $\theta \in (0,1)$,

$$\Phi\left((1 - \theta) \frac{u}{\Phi(x)}\cdot x + \theta\cdot\frac{u}{\Phi(y)}\cdot y\right) > u \; ,$$

since the set $L(u)$ is strictly convex and $z = \left[(1 - \theta) \frac{u}{\Phi(x)}\cdot x + \theta \frac{u}{\Phi(y)}\cdot y\right]$ is an interior point of $L(u)$, implying $\Phi(z) > u$, since if $\Phi(z) = u$ then for some $\theta < 1$, $u \leqq \Phi(\theta z) = \theta\Phi(z) = \theta u$, a contradiction. Take

$$\theta = \frac{\Phi(y)}{\Phi(x) + \Phi(y)}$$

and $\Phi(x + y) > \Phi(x) + \Phi(y)$.

The validity of the restricted law of decreasing returns then follows, because, using the homogeneity, strict additivity and nondecreasing property of $\Phi(\cdot)$,

$$\Phi(\tfrac{1}{2}\cdot\bar{x}) = \tfrac{1}{2}\,\Phi(\bar{x}) \geqq \tfrac{1}{2}\,\Phi(x^{0}) > 0$$

$$\Phi(\tfrac{1}{2}(\bar{x} + 2h)) = \tfrac{1}{2}\,\Phi(\bar{x} + 2h) \geqq \tfrac{1}{2}\,\Phi(\bar{x}) > 0 \; ,$$

and

$$\Phi(\bar{x} + h) = \Phi(\tfrac{1}{2}\bar{x} + \tfrac{1}{2}(\bar{x} + 2h)) > \tfrac{1}{2}\,\Phi(\bar{x}) + \tfrac{1}{2}\,\Phi(\bar{x} + 2h) \; .$$

whence

$$\Phi(\bar{x} + 2h) - \Phi(\bar{x} + h) < \Phi(\bar{x} + h) - \Phi(\bar{x}) ,$$

or, since $\bar{x} = x^o + (N - 2)h$,

$$\Phi(x^o + N \cdot h) - \Phi(x^o + (N - 1)h) < \Phi(x^o + (N - 1)h) - \Phi(x^o + (N - 2)h)$$

for $N \geq 2$.

Note that for any x and y such that $\Phi(x) > 0$ and $\Phi(y) > 0$, and $\theta \in (0,1)$, the production function is strictly concave, since it is strictly super additive and homogeneous.

Thus, it is seen that, if boundedness of the efficient subsets $E(u)$ for $u > 0$ is discarded and one assumes that the technology is positively homogeneous of degree one with strictly convex input sets $L(u)$ for $u > 0$, the technology will obey a law of strictly decreasing product increments over the whole range, no matter which inputs are fixed and which are incremented, explaining Eichhorns result and the properties of pseudo production functions like the Cobb-Douglas and CES.

The reasons why boundedness of the efficient subsets $E(u)$ is deleted and each factor of production is essential, when the production possibility sets $L(u)$ are strictly convex for $u > 0$, are explained by the following proposition.

<u>Proposition 13</u>: A production possibility set $L(u)$ is strictly convex for $u > 0$ if and only if each factor of production is essential and the efficient subset $E(u)$ is unbounded.

If a factor, say the first, is nonessential then there exists an input vector $(0, x_2, \ldots, x_n)$ such that $(0, x_2, \ldots, x_n) \in L(u)$. This vector belongs to the

boundary of $L(u)$ and likewise all input vectors $(0, \lambda x_2, \ldots, \lambda x_n)$ for $\lambda > 1$ belong to the boundary of $L(u)$, implying that $L(u)$ is not strictly convex. If $E(u)$ is bounded, then for any factor of production, say the first,

$$\begin{aligned} \operatorname{Min} \{x_1 \mid x \in L(u)\} &= \operatorname{Min} \left\{x_1 \mid x \in \left(\bar{E}(u) + R^n_+\right)\right\} \\ &= \operatorname{Min} \{x_1 \mid x \in \bar{E}(u)\} \end{aligned}$$

exists, where $\bar{E}(u)$ is the closure of $E(u)$, since $\bar{E}(u)$ is a bounded and closed set. Let x^* yield this Min. Then $x^* \in$ Boundary $L(u)$, since $\bar{E}(u) \subset L(u)$ because $L(u)$ is a closed set and there does not exist any $\varepsilon > 0$ such that $\{x \mid \|x - x^*\| < \varepsilon\} \subset L(u)$ because $x \notin L(u)$ if $x_1 < x_1^*$, and all input vectors $\left(x_1^*, \lambda x_2^*, \ldots, \lambda x_n^*\right)$ for $\lambda > 1$ belong to the boundary of $L(u)$, implying that $L(u)$ is not strictly convex.

Now suppose that $L(u)$ is strictly convex. Then clearly, each factor of production is essential. Also, if

$$\sum_1^n p_i x_i = \alpha, \quad p = (p_1, \ldots, p_n) \geq 0, \quad \alpha > 0$$

is a supporting hyperplane of $L(u)$, it contacts the set $L(u)$ at a unique point $x^*(p)$ of $L(u)$. For $p_1 > 0$ and $p_2 = p_3 = \ldots = p_n = 0$,

$$\operatorname*{Inf}_x \{p \cdot x \mid x \in L(u)\} = \operatorname*{Inf}_x \left\{p \cdot x \mid x \in \left(\bar{E}(u) + R^n_+\right)\right\} = \operatorname*{Inf}_x \{p \cdot x \mid x \in \bar{E}(u)\}$$

does not occur for a bounded $x^* \in \bar{E}(u)$, because then all points $\left(x_1^*, \lambda x_2^*, \ldots, \lambda x_n^*\right)$ likewise belong to $L(u)$ and mor over these points belong to the boundary of $L(u)$, implying that $L(u)$ is not strictly convex. Hence the set $L(u) = \left(\bar{E}(u) + R^n_+\right)$ is not supported by a hyperplane $x_1 = \alpha$ at a finite point $x^*(p)$ for $p = (1, 0, 0, \ldots, 0)$, and the efficient subset $E(u)$ is not bounded.

Eichhorn's [5] assumptions reduce to:

(a) $\Phi(\lambda x) = \lambda\Phi(x)$, $\lambda \geqq 0$, $x \in R^n_+$.

(b) $\Phi(\lambda x_1, \ldots, \lambda x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = \lambda^{r_i} \cdot \Phi(x)$,

$i \in \{1,2, \ldots, n\}$, $\lambda \geqq 0$, $0 < r_i < 1$.

For $\lambda > 0$, $i \in \{1,2, \ldots, n\}$ and $x \in R^n_+$,

$$\Phi\left(\lambda\left(\frac{x_1}{\lambda}\right), \ldots, \lambda\left(\frac{x_{i-1}}{\lambda}\right), \left(\frac{x_i}{\lambda}\right), \lambda\left(\frac{x_{i+1}}{\lambda}\right), \ldots, \lambda\left(\frac{x_n}{\lambda}\right)\right)$$

$$= \Phi\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda}, x_{i+1}, \ldots, x_n\right)$$

$$= \lambda^{r_i} \Phi\left(\frac{x}{\lambda}\right) = \frac{1}{(\lambda)^{1-r_i}} \cdot \Phi(x) ,$$

and, for $\lambda \to +\infty$, it follows that

$$\Phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0 \text{ , } i \in \{1,2, \ldots, n\} ,$$

since the production function is a continuous, concave function for $x \in R^n_+$ when it is homogeneous of degree one. Thus, each factor of production is implied to be essential. Moreover, for $x > 0$,

$$\Phi\left(\frac{x_1}{x_1}, x_2, \ldots, x_n\right) = \Phi(1, x_2, \ldots, x_n) = \frac{1}{(x_1)^{1-r_1}} \cdot \Phi(x)$$

$$\Phi\left(1, \frac{x_2}{x_2}, x_3, \ldots, x_n\right) = \Phi(1,1,x_3, \ldots, x_n) = \frac{1}{(x_2)^{1-r_2}} \cdot \Phi(1, x_2, \ldots, x_n)$$

$$= \frac{1}{(x_1)^{1-r_1}} \frac{1}{(x_2)^{1-r_2}} \cdot \Phi(x) ,$$

and, continuing in this fashion, one obtains

$$\Phi(1,1, \ldots, 1) = \frac{1}{\prod_{1}^{n} (x_i)^{(1-r_i)}} \cdot \Phi(x) .$$

Due to Property (a) it follows that:

$$\Phi(x) = \Phi_o \cdot \prod_{1}^{n} x_i^{\nu_i} , \sum_{1}^{n} \nu_i = 1 , 0 < \nu_i < 1 , i \in \{1,2, \ldots, n\} ,$$

where $\Phi_o = \Phi(1,1, \ldots, 1)$ and $\nu_i = (1 - r_i)$. Thus, his assumptions imply that the production function is a Cobb-Douglas production function with strictly convex level sets (production possibility sets), which is a special case of a positively homogeneous technology (degree one) with strictly convex production possibility sets, just as is the CES production function, both of which violate an essential property of a technology, i.e., boundedness of the efficient subset for any positive output rate. The proposition described above as the restricted law of diminishing returns encompasses all cases of this kind.

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14 KEY WORDS	LINK A		LINK B		LINK C	
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E R R A T A

PROOF OF THE LAW OF DIMINISHING RETURNS

by

Ronald W. Shephard
ORC 69-37
December 1969

Typographical Errata:

1) Pg. 35, line 4: replace period by comma and change T in Then to lower case.

2) Pg. 43, line 3: insert λ as multiplier for $x_{i+1}, \ldots, x_n$ in the arguments of Φ.

3) Pg. 28, last line: $\bigcap_{0 \leqq u < u^0} L^0(u)$.

Footnote Insertion: For Property P.10*, Pg. 6

*This property is not required for the arguments to follow.

Correction: Pg. 25, lines 5 and 6 should read.

Then by Property A.3 it follows, ...